FURTHER EXAMINATION OF
OPTIMAL FIRE SUPPORT STRATEGIES

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NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

FURTHER EXAMINATION OF

OPTIMAL FIRE SUPPORT STRATEGIES

by

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A Further Examination of

Optimal Fire Support Strategies

by

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ABSTRACT

Previous thesis research has studied optimal time-sequential artillery fire distribution strategies in an attack scenario by considering a two-person, zero-sum, deterministic differential game. This past work is reviewed and extended by developing additional computational considerations (i.e., manifold of discontinuity for the dual variables) and incorporating these into an existing digital computer program for the determination of optimal time-sequential fire support strategies. The military significance and implications of the structure of an optimal fire distribution policy are discussed.

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NOTATION

$X_1(t)$, $X_2(t)$, $Y_1(t)$, $Y_2(t)$ -- State variables, number of survivors of X_1, X_2, Y_1, Y_2 at time t.

T -- Specified duration of the "approach to contact".

$p_1(t)$, $p_2(t)$, $q_1(t)$, $q_2(t)$ -- Adjoint (dual) variables, the value of X_1, X_2, Y_1, Y_2 respectively at time t.

a_1 -- Attrition rate coefficient, average rate at which are Y_2 attrits X_1.

a_2 -- Average rate at which one Y_2 attrits X_2.

c -- Average rate at which one Y_1 attrits X_1.

b_1 -- Average rate at which one X_2 attrits Y_1.

b_2 -- Average rate at which one X_2 attrits Y_2.

v -- Decision (control) variable, fraction of Y_2 directed at X_1, $0 \leq v \leq 1$.

u -- Fraction of X_2 directed at Y_1, $0 \leq u \leq 1$.

v^*, u^* -- Optimal values of v and u.

J -- Criterion functional (objective function).

τ -- Backwards time, $\tau = T - t$.

I. INTRODUCTION AND PURPOSE

The subject of fire support allocation in military conflict has received considerable attention during the last few years, particularly in the area of optimizing the distribution of limited fire support resources. One specific example of the application of optimization techniques to such allocation problems is a paper by Kawara [1] in the field of differential games. In an extension of the work of Weiss [2], he examined the optimal strategies of two heterogeneous forces in a scenario in which each force consisted of a combat unit (such as infantry) and a support unit (such as artillery) wherein the strategy of each force was the distribution it made of its supporting fires between the other's infantry and artillery. Kawara concluded from his analysis that, given the artillery units of each opponent are never reduced to zero, the optimal strategies of the opponents are independent of force levels.

Taylor [3] showed, for a given set of combat attrition equations such as used by Kawara, that optimal strategies are a function of the criterion functional (objective function), and that he (Kawara) picked the only type of criterion functional -- the ratio of combatants at the end of the engagement -- which produced optimal strategies completely independent of force levels. Taylor also examined a modified version of Kawara's model in order to show that not only can the optimal strategy of one of the combatants depend directly

on the enemy force levels, but also that the optimal strategy of that side is to sometimes split its supporting fires between the enemy's infantry and artillery.

Ellis [4] considered in his thesis this same two-person, zero-sum differential game and its solution procedure (as outlined by Taylor) which consisted of synthesizing extremal trajectories[1] using a backwards sweep process. For his thesis he constructed a lengthy computer program which numerically implemented the backwards sweep process and allowed him to flood the state space with extremal trajectories. As an output from his program, Ellis created a two-dimensional computerized plot which most vividly illuminates the dependence of strategy on force levels in Taylor's model.

However, while examining several cases of Taylor's scenario, Ellis found that, for several combinations of parameters and values of state variables used to initialize the backwards sweep process, it was not possible to reach certain beginning values of the state space. This lack of trajectories passing through certain portions of the five-dimensional state space was coined as a "void" by Taylor. Taylor [3] subsequently provided an analytical explanation for the existence of the "void" discovered by Ellis, and developed a candidate solution for filling it with optimal trajectories.

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[1] A trajectory generated by a realization of a strategy which satisfies the necessary conditions of optimality for all points in time.

The purpose of this thesis was to incorporate Taylor's solution for filling the "void" into the computational routine developed by Ellis, to numerically solve the differential game with the modified routine, and to demonstrate with graphical plot the effect of Taylor's solution for filling the "void".

II. DEVELOPMENT OF OPTIMAL TIME-SEQUENTIAL FIRE SUPPORT STRATEGIES

A. THE COMBAT SCENARIO

The combat scenario examined by Taylor [3] (extending previous work by Kawara [1] and Weiss [2]) consisted of two opposing heterogeneous forces in a fixed duration "approach to contact" wherein one's infantry, X_1, is attacking the other's infantry, Y_1. The game terminates at the moment that contact (close combat) between combatants occurs (after a fixed time, T), whereupon artillery fires must be shifted from the enemy's infantry in order to preclude the inflicting of casualties on one's own infantry. Figure 1 depicts the situation examined by Taylor.

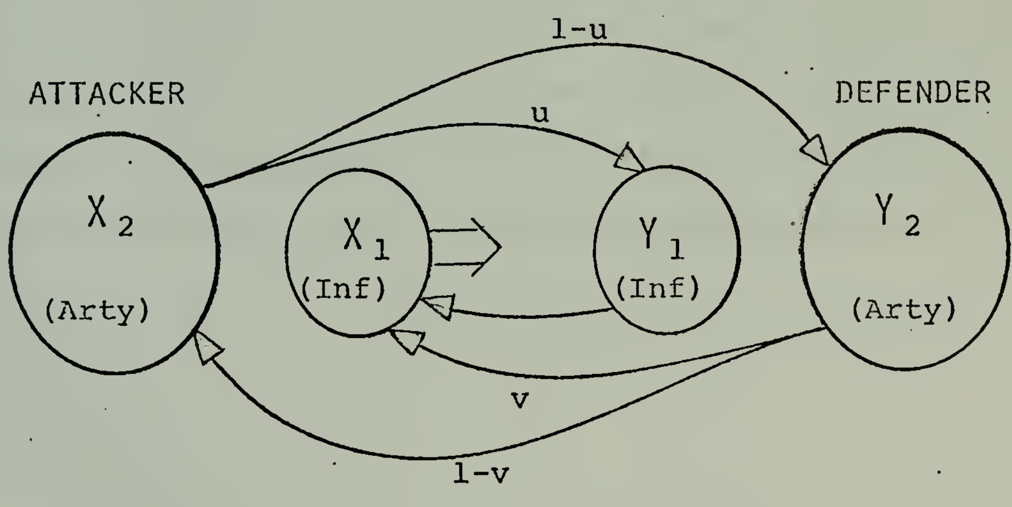

Figure 1. The Tactical Situation

As can be seen, each side's artillery can distribute its fires with complete latitude at its enemy's infantry and/or artillery. All attrition caused by artillery fires was assumed by Taylor to follow a "linear-law" attrition process, except for the defender's artillery, Y_2, against the attacker's artillery, X_2, which was formulated as following a "square-law" attrition process in as much as it was felt the defender has superior target acquisition capability. Due to the condition that the attacker's infantry, X_1, is moving toward the defender's infantry, Y_1, it was assumed that the Y_1 fires on X_1 followed a "linear-law" attrition process, reflecting the difficulty of firing aimed fire at a moving target. One should also notice in Figure 1 that X_1 has been pictured as placing no fires on Y_1, so assumed because of the difficulty of delivering effective fire while in rapid movement to contact. The objective of both opponents during the "approach to contact" is to maximize the ratio of its infantry to its opponent's infantry at the moment the "approach to contact" phase is ended and "close combat" begins.

B. DIFFERENTIAL GAME FORMULATION

Taylor [3] modeled the scenario in section II.A. as a two-person, zero-sum differential game as follows:

$$\underset{U}{\text{MAXIMIZE}} \quad \underset{V}{\text{MINIMIZE}} \quad J = X_1(T)/Y_1(T) \text{ [2]}$$

Subject To:

$$dX_1/dt = -va_1X_1Y_2 - cX_1Y_1 \qquad (=f_1)$$

$$dX_2/dt = -(1-v)a_2Y_2 \qquad (=f_2)$$

$$dY_1/dt = -ub_1Y_1X_2 \qquad (=g_1)$$

$$dY_2/dt = -(1-u)b_2Y_2X_2 \qquad (=g_2)$$

$$0 \leq u(t) \leq 1 \quad 0 \leq v(t) \leq 1$$

$$\left.\begin{array}{l} 0 \leq Y_i(t) = Y_i \\ 0 \leq X_i(t) = X_i \end{array}\right\} \quad i=1,2 \qquad \text{(SVIC's)}$$

Initial Conditions: $X_i(t=0) = X_i^0$, $Y_i(t=0) = Y_i^0$, $i=1,2$

It is quite evident that the state variable inequality constraints (SVIC's) hold for all the state variables (X_i, Y_i, $i=1,2$) except for X_2 which can take on negative values, mathematically speaking; however, for the purposes of his examination, Taylor considered X_2 to remain strictly positive, thus by-passing any need for consideration of the problems generated by SVIC's, such as discussed in Speyer [5].

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[2] In general $J = J(t,\underline{X},\underline{Y}, u,v) = G\{\underline{X}(T),\underline{Y}(T)\} + \int_t^T L\{s,\underline{X}(s),\underline{Y}(s),u(s),v(s)\}ds$, but for Taylor's scenario, $J = G\{\underline{X}(T),\underline{Y}(T)\} + 0 = X_1(T)/Y_1(T)$.

C. DEVELOPMENT OF SOLUTION

In generating the solution to the differential game presented in section II.B., Taylor applied a method similar in form to that used in optimal control problems, utilizing the "Maximum Principle" as applied to differential games such as in [6] and the theory of singular extremals. At this time, it should be noted that the strategies developed by Taylor were not shown to satisfy sufficient conditions for optimality, to do so being beyond the scope of the current investigation.

The general steps followed by Taylor in generating the extremal controls to the differential game in section II.B. are outlined below:

1. Introduce the dual (adjoint) variables $p_1(t)$, $p_2(t)$, $q_1(t)$, $q_2(t)$ which correspond to the state variables $X_1(t)$, $X_2(t)$, $Y_1(t)$, $Y_2(t)$, respectively.

   [$p_i(t) = \partial W(t,\underline{X},\underline{Y})/\partial X_i(t)$

   where $W(t,\underline{X},\underline{Y})$ is differentiable and

   $$W = (t,\underline{X},\underline{Y}) = W\{t=0,\underline{X}^0,\underline{Y}^0;\ t=T,\underline{X}(T),\underline{Y}(T)\}$$

   $$= \underset{u(t_0,T)}{\text{MAXIMUM}}\ \underset{v(t_0,T)}{\text{MINIMUM}}\ J = X_1(T)/Y_1(T)]$$

2. Form the Hamiltonian H:

   $$H(t,\underline{X},\underline{Y},u,v) = L(t,\underline{X},\underline{Y},u,v) + \sum_{i=1}^{2}(p_i f_i + q_i g_i)$$

   $$= \sum_{i=1}^{2}(p_i f_i + q_i g_i).$$

3. Determine the extremal strategic variable pair (extremal controls), (u^*, v^*), by maximize-minimizing the Hamiltonian.

4. Construct the system of time history equations (canonical equations):

$$dX_i/dt = \partial H(t,\underline{X},\underline{Y},\underline{p},\underline{q},u^*,v^*)/\partial p_i(t)$$
$$dY_i/dt = \partial H(t,\underline{X},\underline{Y},\underline{p},\underline{q},u^*,v^*)/\partial q_i(t)$$
$$dp_i/dt = -\partial H(t,\underline{X},\underline{Y},\underline{p},\underline{q},u^*,v^*)/\partial X_i(t)$$
$$dq_i/dt = -\partial H(t,\underline{X},\underline{Y},\underline{p},\underline{q},u^*,v^*)/\partial Y_i(t)$$
$$\text{for } i=1,2$$

5. Determine the terminal boundary conditions for the adjoint variables in preparation for synthesizing the extremals in backwards time (τ):

$$p_i(t=T) = p_i(\tau=0) = \partial G\{\underline{X}(T),Y(T)\}/\partial X_i(T)$$
$$= \partial\{X_1(T)/Y_1(T)\}/\partial X_i(T)$$
$$q_i(t=T) = q_i(\tau=0) = \partial G\{\underline{X}(T),Y(T)\}/\partial Y_i(T)$$
$$= \partial\{X_1(T)/Y_1(T)\}/\partial Y_i(T)$$
$$\text{for } i=1,2$$

6. Compute the amount of discontinuous jump made by the adjoint variables across a manifold of discontinuity of both u^* and v^*, simultaneously. A "void" in the state space occurs only in those cases where a discontinuity in the adjoint variables exists. (Alternatively speaking, this final step covers the procedure for filling the "void", as developed by Taylor [3] after the discovery of the existence of the "void" by Ellis [4].)

For a detailed explanation of these six steps, see Taylor [3].

III. FILLING THE "VOID"

A. MODIFICATION OF THE COMPUTATIONAL ROUTINE

The computer program developed by Ellis [4] which numerically synthesized the extremal controls to the differential game in section II.B. using Taylor's original (before the discovery of the state space "void") solution, consisted of 546 lines of Fortran IV coding. Generally, the program encompassed steps 1 through 5 listed in section II.C.

In order to make an alteration to Ellis' program (to implement the filling of the "void", step 6 in section II.C.) without inadvertently and erroneously changing the basic logic scheme, it was first necessary to develop a flowchart outlining the logic used by Ellis.

A detailed flowchart showing all the intricacies of methods used in the program, such as the Runge-Kutta fourth-order numerical integration algorithm or the Newton-Raphson routine for numerically solving transcendental equations, would have been of no practical use; therefore, a block-type flowchart showing the inter-relationships between the critical segments of the backwards sweep process was drawn to model the overall procedure of extremal synthesis and graphical plotting. Key flag and count variables were included in the diagram as well as all significant logic nodes.

An immediate spinoff from the completed flowchart was to catch several errors in Ellis' train of logic which had not been previously detected during initial program operation

and debugging. Once Taylor's differential game solution was clearly understood, and Ellis' program steps were associated one-to-one with the solution steps, the incorporation of Taylor's solution for filling the "void" into Ellis' program was relatively uncomplicated, ultimately requiring only about 50 lines of Fortran IV coding.

Lastly, fine tuning of the numerical integration time steps affected by the modification was necessary in order to insure acceptable error level in the numerical approximations. (The verification of the numerical integration coding was previously accomplished by Ellis, and, therefore, not repeated here.)

B. EXAMPLE OF FILLING THE "VOID"

Figure 2 is a graphical plot, done on an IBM 360 computer, of a specific case of the differential game examined by Taylor in which a discontinuity in the adjoint (dual) variables existed. Note the dearth of trajectories through a large portion of the two-dimensional state space (Z,t)[3], where t occurs during the initial stages of the approach to contact. The absence of extremal trajectories passing through those portions of the state space is what has been called the "void".

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[3] $Z = Y_1(t)/Y_2(t)$ was picked by Ellis [4] as a suitable transformed state variable to demonstrate the force level dependence of optimal strategies.

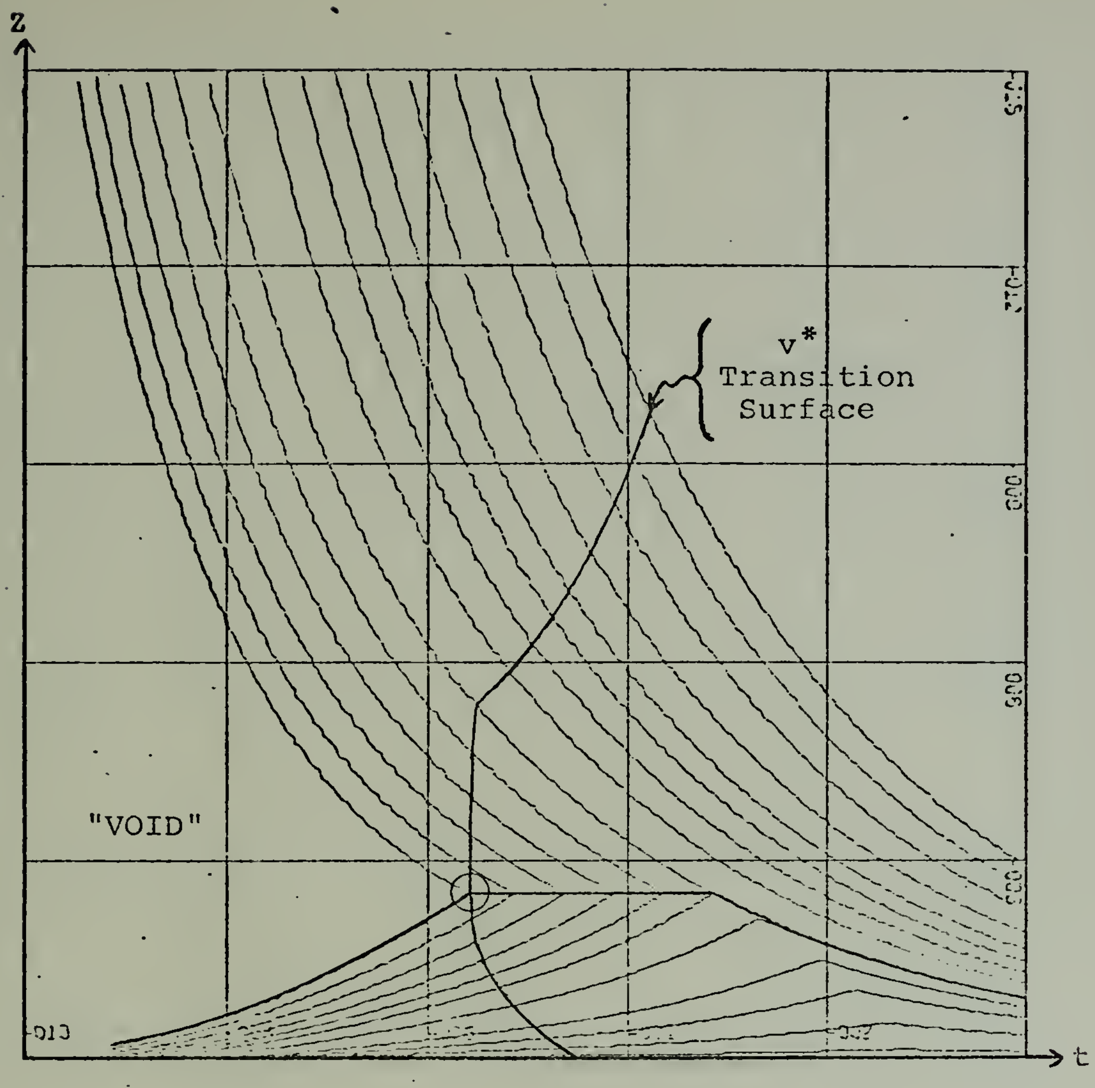

Figure 2. The "Void"

Moreover, once the modification to Ellis' computer program (incorporating the discontinuous jump of the adjoint variables p and q at the manifold of discontinuity of u* and v*) was applied, this same "void" was subsequently flooded by optimal trajectories as graphically displayed in Figure 3.

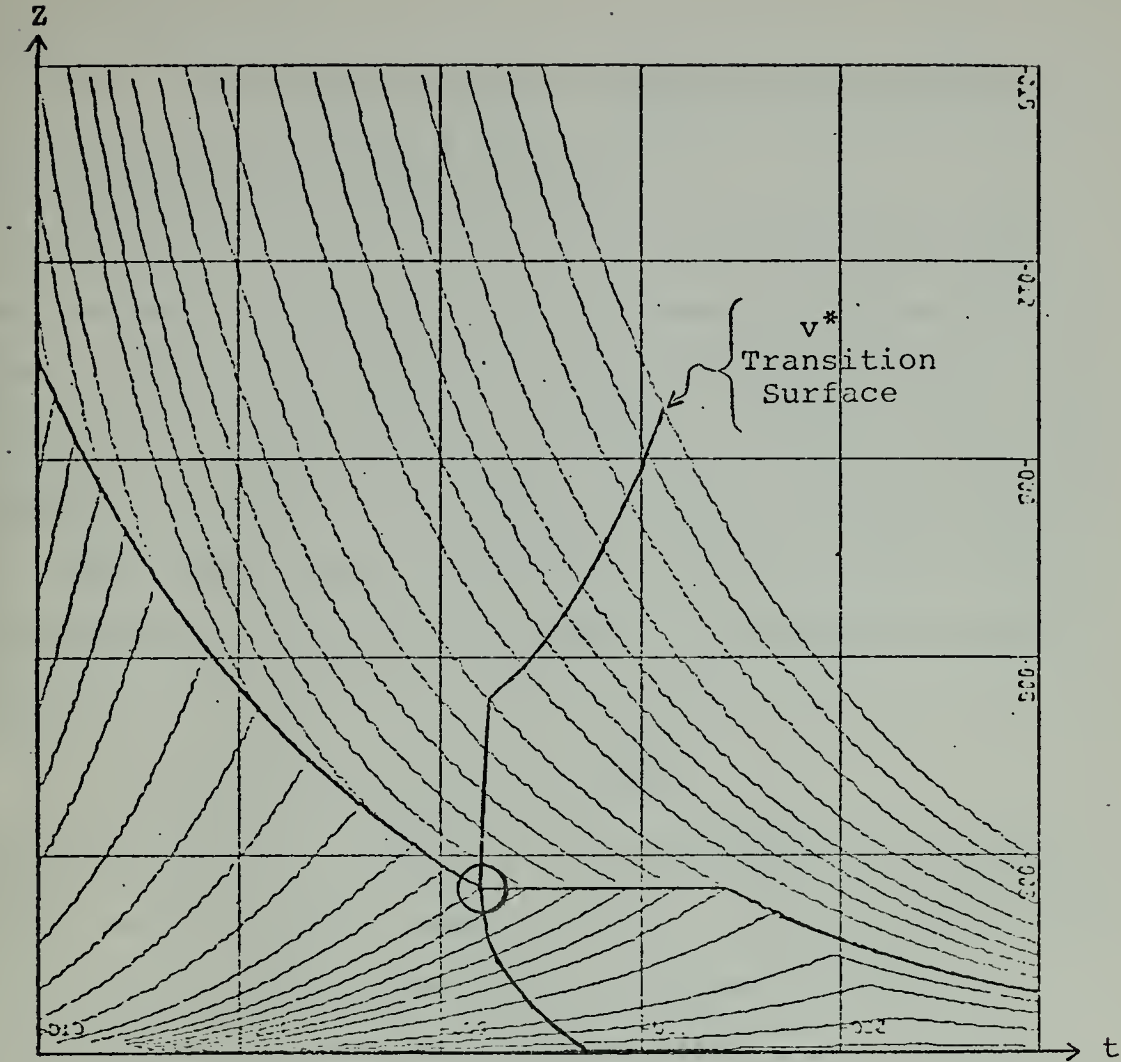

Figure 3. The Filled "Void"

To explain Figures 2 and 3 further, the discontinuity of v* occurs on the v* transition surface, and the discontinuity of u* on the u* transition surface, depicted by the dark extremal trajectory in Figure 3. The intersection of these two transition surfaces, indicated by a circle in both Figures 2 and 3, is the point in backwards time where the "void" occurs, as clearly seen in Figure 2.

IV. SIGNIFICANCE OF THE MODEL AND ITS SOLUTION

A. INTERPRETATION OF THE SOLUTION

The first objective of Taylor's [3] work was to show that optimal strategies are not necessarily independent of force levels; this Ellis [4] aptly illustrated by his plotting the transformed state variable, Z, against time, which indicated the shift of strategy in conjunction with particular enemy force levels. Even more vividly, under certain circumstances in Taylor's solution, the following equation holds:

$$u = \{b_2/(b_1+b_2)\}(1-q_2Y_2/p_2X_2)$$

which demonstrates explicitly the dependence of strategy on force levels, in this case Y_2 and X_2. In mathematical terms, the strategies are closed-loop strategies in that $U = U\{t,\underline{X}(t),\underline{Y}(t)\}$ and $V = V\{t,X(t),Y(t)\}$, or, in other words, the strategies of the opponents are feedback dependent. In military terms, the significance of these results is that intelligence concerning the enemy force levels is extremely important in the development of one's strategy. Because of the uncertainty inherent in intelligence estimates in the real world, Taylor's results imply that one's strategy must be continuously reevaluated as a result of the changing intelligence picture.

The second objective that Taylor sought to accomplish was to illustrate the fact that supporting fires may be

split between opposing forces. There exists in the solution to Taylor's model specific instances when the optimal strategy is to fire on the enemy's two different forces proportional to their relative strengths; or, mathematically speaking,

$$u = b_2/(b_1+b_2), \quad 1-u = b_1/(b_1+b_2)$$

such that they are attrited at proportionally the same rate, keeping $Z = Y_1(t)/Y_2(t)$ constant. This is well illustrated in the graphical plot developed by Ellis when an extremal remains at a constant level for a finite period of time as seen between t* and t** in Figure 4, picturing a single optimal trajectory extracted from Figure 3.

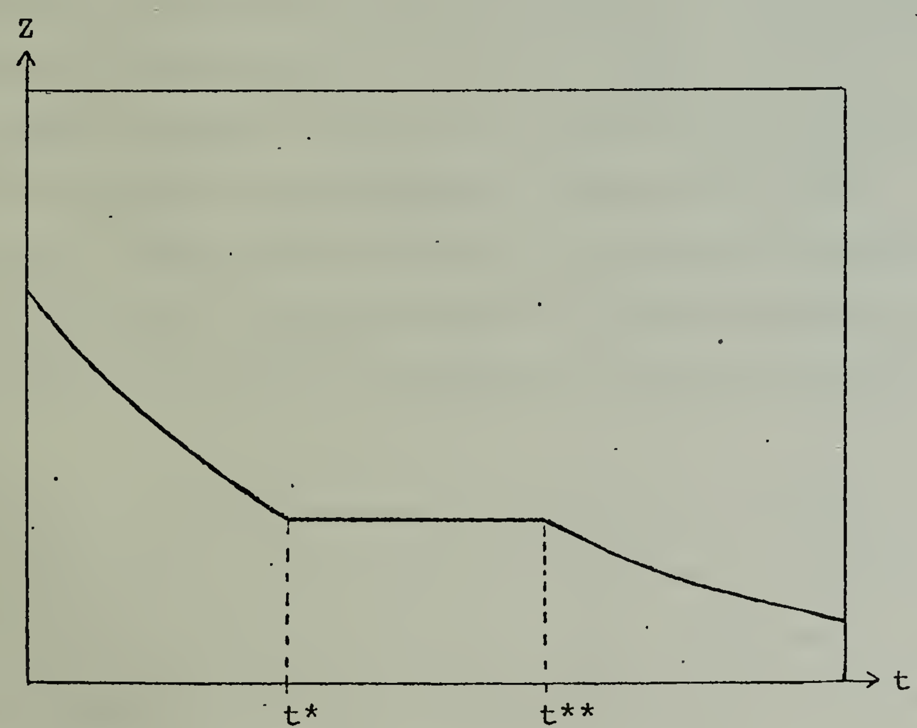

Figure 4

Notwithstanding, the most significant lesson that can be learned from the combined results of this work, Taylor [3] and Ellis [4] is aimed not at the military decision maker, but at the analyst who is attempting to model combat dynamics: in a differential game, the form of the dynamics used to depict a situation may impose unacceptable characteristics on the strategies of the game players, causing them to violate very basic concepts from the real world the analyst is attempting to model, such as in the case of Kawara [1].

B. FUTURE CONCERNS

There appears to be a logical extension of this work that could be made, possibly to satisfy a Master's Thesis requirement, beginning with the assignment of appropriate constants of dimensionality to Taylor's model, such an undertaking being necessary before the military artist can effectively interpret the strategy or tactics that the differential game solution dictates. Once this is accomplished, of further interest would be a sensitivity analysis on the model, using Ellis' program as modified by this thesis.

For the interested reader, the resultant program, flowchart, and computer cards from this work may be obtained from Associate Professor James G. Taylor, Department of Operations Research and Administrative Sciences, Naval Postgraduate School, Monterey, California 93940.

V. SUMMARY

The method Taylor [3] derived for filling the "void" found in his original solution to the modified version of Kawara's [1] differential game has been found to be successful. Such was shown to be true by means of incorporating the aforementioned method into Ellis' [4] computer routine for numerically executing Taylor's original solution, and then looking at "before" and "after" snapshots of a case involving a "void".

The force level dependence of optimal strategies, the splitting of supporting fires, and the sensitivity of strategy to model attrition dynamics has also been discussed.

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